Fourier transform of an impulsion train

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This document was inspired by http://math.ut.ee/~toomas_l/harmonic_analysis/ and gives the demonstration of the following theorem:

**Theorem 1 (Impulsion train).** The Fourier transform of a spatial domain impulsion train of period $T$ is a frequency domain impulsion train of frequency $\Omega = 2\pi / T$.

$$\sum_{p \in \mathbb{Z}} \delta(x - pT) \leftrightarrow \Omega \sum_{k \in \mathbb{Z}} \delta(x - k\Omega)$$  (1)

**Reminders**

**Fourier Coefficients**

Let $f$ be a $T$-periodic function, we have:

$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ik\Omega x}$$

with

$$\begin{align*}
\Omega &= \frac{2\pi}{T} \\
c_k &= \frac{1}{T} \int_0^T f(t) e^{-ik\Omega t} dt
\end{align*}$$

The $c_k$ are called the Fourier coefficients of $f$. This coefficient can be rewritten as an integral over any interval of length $T$. In particular, we will use:

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-ik\Omega t} dt$$  (2)

**Proof.** Let $g(t) = f(t) e^{-ik\Omega t}$. It is a $T$-periodic function since we have:

$$g(t + T) = f(t + T) e^{-ik\Omega(t+T)} = f(t) e^{-ik\Omega t} e^{-ik\Omega T}$$

$$= f(t) e^{-ik\Omega t} e^{-i2\pi} = g(t)$$

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Equation (2) is said due to the fact that the integral of a $T$-periodic function is constant over any interval of length $T$ as can be seen from:

$$
\int_{c}^{c+T} g(t) dt = \int_{c}^{0} g(t) dt + \int_{0}^{T} g(t) dt + \int_{T}^{c+T} g(t) dt = \int_{c}^{0} g(t) dt + \int_{0}^{T} g(t) dt + \int_{0}^{T} g(t+u) du = \int_{0}^{T} g(t) dt
$$

Fourier Transform

The **Fourier transform** $F(\omega)$ of a real-valued function $f(x)$ is defined by:

$$
F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx
$$

The **inverse Fourier transform** is given by the relation

$$
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega x} d\omega
$$

When two functions are related by the Fourier transform, we note:

$$
f(x) \overset{FT}{\rightarrow} F(\omega)
$$

We have the **symmetry property**:

$$
\text{if } f(x) \overset{FT}{\rightarrow} F(\omega) \text{ then } F(x) \overset{FT}{\rightarrow} 2\pi f(-\omega)
$$

and the **linearity property**:

$$
\begin{cases}
  f(x) \overset{FT}{\rightarrow} F(\omega) & \text{then } (\lambda f + g)(x) \overset{FT}{\rightarrow} (\lambda F + G)(\omega) \\
  f(x) \overset{FT}{\rightarrow} F(\omega)
\end{cases}
$$

The Dirac impulsion

The Dirac function $\delta(x)$ has the **sifting property**. If $f$ is continuous at point $a$:

$$
\int_{-\infty}^{\infty} f(t)\delta(t-a) dt = f(a)
$$
The Fourier transform of a translated Dirac is a complex exponential:
\[
\delta(x - a) \xrightarrow{FT} e^{-ia\omega}
\]  

(8)

**Impulsion train**

Let’s consider \( it(x) = \sum_{p \in \mathbb{Z}} \delta(x - pT) \) a train of \( T \)-spaced impulsions and let’s compute its Fourier transform. We first rewrite \( f \) using its Fourier coefficients:
\[
it(x) = \sum_{k \in \mathbb{Z}} c_k e^{ik\Omega x}
\]

where \( \Omega = 2\pi / T \). Using Eq. (2), we have:
\[
c_k = \frac{1}{T} \int_{-T/2}^{T/2} it(t)e^{-ik\Omega t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t - pT)e^{-ik\Omega t} dt
\]
\[
= \frac{1}{T} \sum_{p \in \mathbb{Z}} \int_{-T/2}^{T/2} \delta(t - pT)e^{-ik\Omega t} dt
\]

Since the function \( t \mapsto \delta(t - pT) \) is null over the interval \([-T/2, T/2] \) for \( p \neq 0 \), we are left with only one term in the summation:
\[
c_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)e^{-ik\Omega t} dt = \frac{1}{T} \int_{-\infty}^{\infty} \delta(t) e^{-ik\Omega t} dt = \frac{1}{T} \underbrace{e^{-i\omega 0}}_{=1} \text{ by (7)}
\]

So finally we have an expression of the impulse train:
\[
it(x) = \frac{1}{T} \sum_{k \in \mathbb{Z}} e^{ik\Omega x}
\]

(9)

Applying the symmetry property (5) to the Fourier transform of a Dirac (8) we find:
\[
e^{-ik\Omega(-x)} \xrightarrow{FT} 2\pi \delta(-(-x) - k\Omega)
\]
\[
e^{ik\Omega x} \xrightarrow{FT} 2\pi \delta(x - k\Omega)
\]

Applying linearity (6) to expression (9) we finally get the equality (1).