A Theoretical Analysis of Compactness Of the Light Transport Operator

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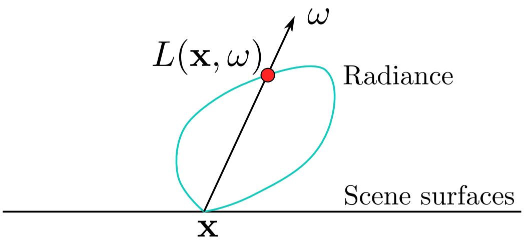
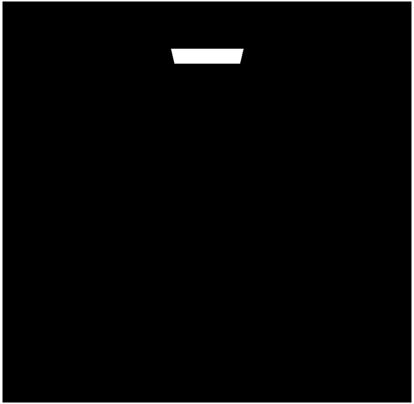
Context / motivation

Light Transport Operator:

$$L(\mathbf{x}, \omega) \longmapsto (\mathbf{T}L)(\mathbf{x}, \omega)$$





L_e

Context / motivation

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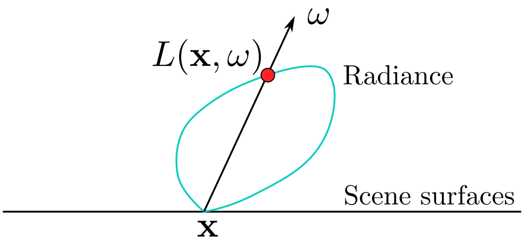

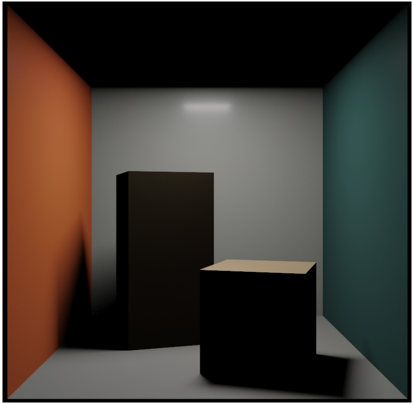

$\mathbf{T}L_e$

Context / motivation

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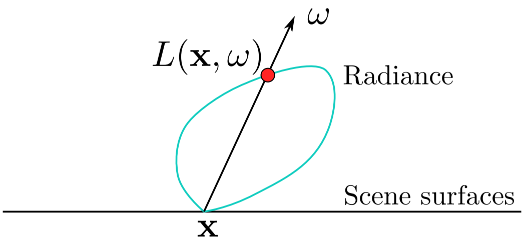
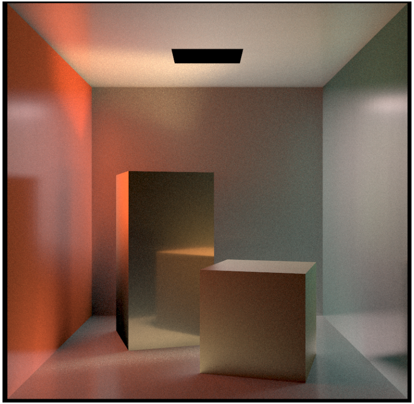
$$L(\mathbf{x}, \omega) \longmapsto (\mathbf{T}L)(\mathbf{x}, \omega)$$



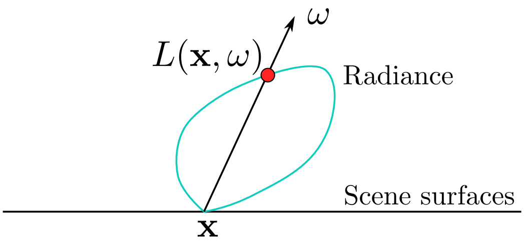


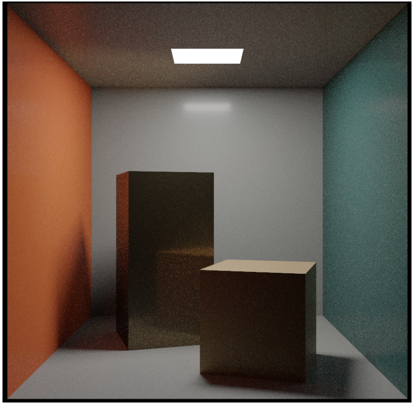
$\mathbf{T}^2 L_e$

Context / motivation

Light Transport Operator:

$$L(\mathbf{x}, \omega) \longmapsto (\mathbf{T}L)(\mathbf{x}, \omega)$$



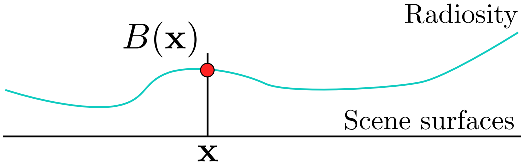


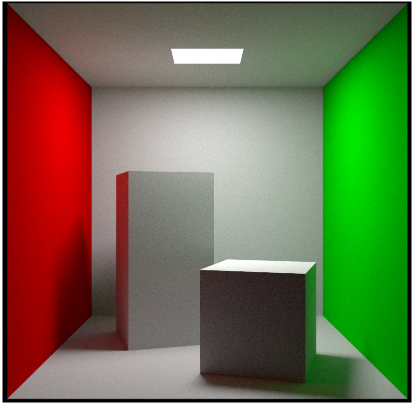
$$L = L_e + \mathbf{T}L_e + \mathbf{T}^2 L_e + \dots$$

Context / motivation

Light Transport Operator:

$$B(\mathbf{x}) \longmapsto (\mathbf{T}_b B)(\mathbf{x})$$





$$L = L_e + \mathbf{T}_b L_2 + \mathbf{T}_b^2 L_e + \ldots$$

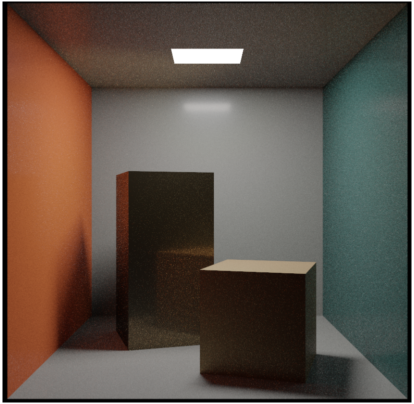
Context / motivation

Light Transport Operator:

$$\underset{\infty-\text{dimensional}}{\mathbf{T}, \mathbf{T}_b} \longrightarrow \underset{n \times n \text{ matrix}}{T_n}$$

Finite Rank approximations of **T**:

- piecewise constant functions [Goral84,Hanrahan91]
- wavelets [Gortler93]
- spherical harmonics [Green2003, Ramamoorthi06]
- polynomials [Ben-Artzi2008]



$$L = L_e + \mathbf{T} L_e + \mathbf{T}^2 L_e + \ldots$$

Context / motivation

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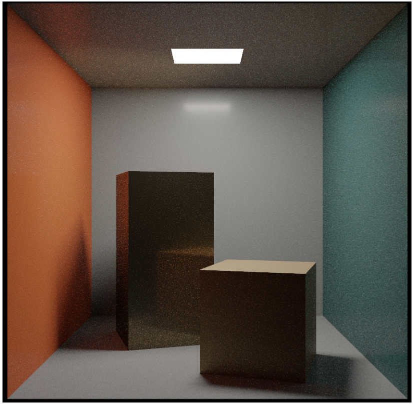
Light Transport Operator:

$$\mathbf{T}, \mathbf{T}_b \longrightarrow T_n$$

$\infty-$dimensional → $n \times n$ matrix

Used in...

- Finite Element Global Illumination



$$L = L_e + \mathbf{T} L_e + \mathbf{T}^2 L_e + \ldots$$

Assumption: for a specific L, $T_n L \to \mathbf{T} L$

Context / motivation

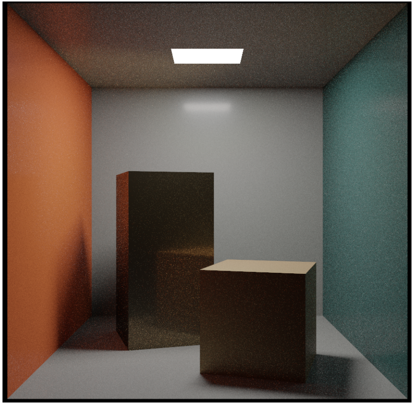
Light Transport Operator:

$$\mathbf{T}, \mathbf{T}_b \longrightarrow T_n$$

∞−dimensional → $n \times n$ matrix

Used in...

- Finite Element Global Illumination
- Precomputed Radiance Transfer
- Neural rendering



$$L = L_e + \mathbf{T}L_e + \mathbf{T}^2 L_e + \ldots$$

Assumption: for all L, $\| T_n L - \mathbf{T}L \| < \epsilon$

Context / motivation

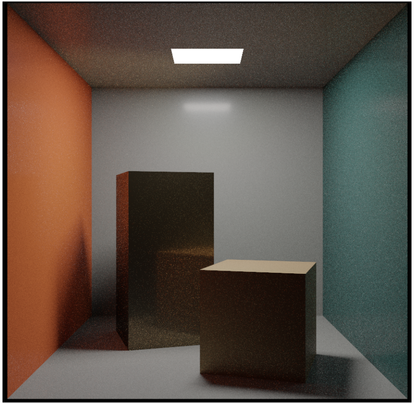
Light Transport Operator:

$$\mathbf{T}, \mathbf{T}_b \longrightarrow T_n$$

$\infty-$dimensional → $n \times n$ matrix

Used in...

- Finite Element Global Illumination
- Precomputed Radiance Transfer
- Neural rendering
- Inverse lighting



$$L = L_e + \mathbf{T} L_e + \mathbf{T}^2 L_e + \ldots$$

Assumption: $\mathbf{T}$ is invertible, and $T_n^{-1} \approx \mathbf{T}^{-1}$

Context / motivation

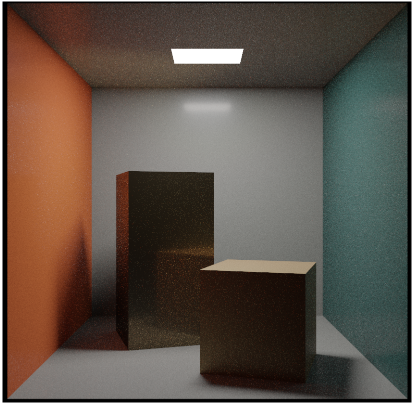
Light Transport Operator:

$$\mathbf{T}, \mathbf{T}_b \longrightarrow T_n$$

$\infty-$dimensional $\qquad$ $n \times n$ matrix

Used in...

- Finite Element Global Illumination
- Precomputed Radiance Transfer
- Neural rendering
- Inverse lighting
- Dimensional analysis



$$L = L_e + \mathbf{T}L_e + \mathbf{T}^2 L_e + \dots$$

Assumption: $\sigma(T_n) \to \sigma(\mathbf{T})$

Context / motivation

General linear operator **T** in infinite dimensions:

- uncountable eigen/singular-values, ∞-multiplicity
  - ☹ $\sigma(T_n) \not\to \sigma(\mathbf{T})$
  - ☹ $\|(T_n - \mathbf{T})L\| \to 0$ depending on L

Context / motivation

Compact operator (See our paper):

- countable eigenvalues converging to 0
  - $\sigma(T_n) \to \sigma(\mathbf{T})$
  - $\|(T_n - \mathbf{T})L\| \to 0$ uniformly

$$\begin{bmatrix} 0.8 & & & \\ & 0.6 & & \\ & & 0.2 & \\ \vdots & & & \ldots \end{bmatrix}$$

- Example: integral operators with square-integrable kernel (e.g. convolution)

Context / motivation

Compact operator (See our paper):

- countable eigenvalues converging to 0
  - ☺ $\sigma(T_n) \to \sigma(\mathbf{T})$
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- Example: integral operators with square-integrable kernel (e.g. convolution)

Are Light Transport Operators $\mathbf{T}$ and $\mathbf{T}_b$ compact ?

Summary of our findings

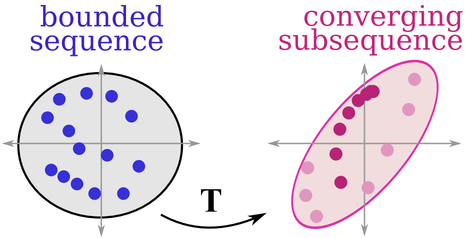
In this paper, we show

- $\mathbf{T}$ is never *compact*
- $\mathbf{T}$ still shares key properties with compact operators
  - in closed scenes, $\mathbf{T}$ has a complete Schmidt expansion (a.k.a. SVD in ∞ dimensions)
- $\mathbf{T}_b$ ($\mathbf{T}$ re-formulated in Lambertian scenes) is generally not *compact*
- $\mathbf{T}_b$ shares key properties with compact operators
  - not invertible
  - acts as a low-pass filter (away from edges)
- local reflectance $\mathbf{K_x}$ is compact
  - $\Rightarrow$ not invertible
- connections to historical choices & future developments.

Lambertian case

How do we prove that $\mathbf{T}_b$ is **not** compact?

"Compact linear operators map bounded sequences into sequences with converging subsequences"

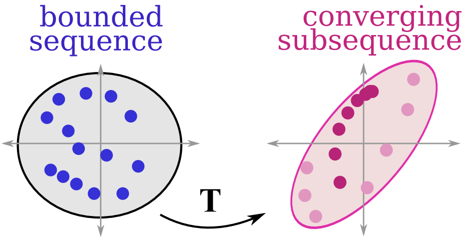


(See paper for more details)

Lambertian case

How do we prove that $\mathbf{T}_b$ is **not** compact?

”Find one bounded sequence $\{B_n\}_{n>0}$, such that $\mathbf{T}_b B_n$ has no converging subsequence”



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Lambertian case

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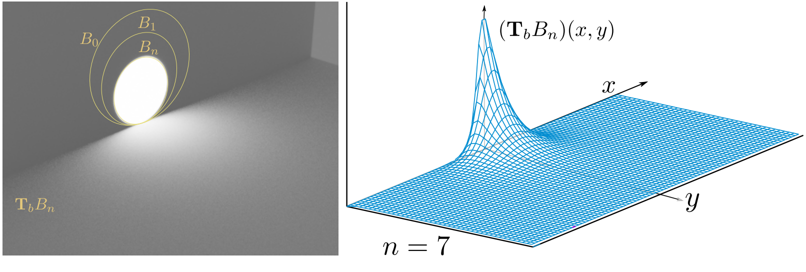


(See paper for more details)

Consequences of non-compactness

No uniform approximation by finite rank operators of **T** and $\mathbf{T}_b$

$\Rightarrow$ meshes provide bounded error **only** for a specific light distribution

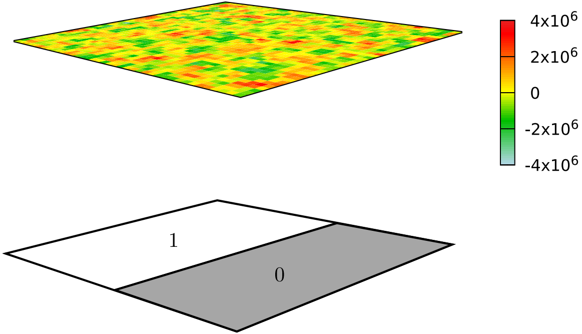
- Adaptive methods are needed and require guidance
- PRT and Neural methods cannot give global error garantee
- Spectral analysis based on matrix approximations need additional justification

$\mathbf{T}_b$ is not the uniform limit of operators with bounded kernels

$\Rightarrow$ bias when connecting close points using a bounded weight in path tracing

Consequences of non-invertibility

- $\mathbf{K_x}$ is not invertible
  $\Rightarrow$ inverse shading is ill-posed
- inverting $\mathbf{T}_b$ looks like de-convolution (See paper)



Consequences of non-invertibility

- $\mathbf{K_x}$ is not invertible
  $\Rightarrow$ inverse shading is ill-posed
- inverting $\mathbf{T}_b$ looks like de-convolution (See paper)
- inverting multi-bounce transport is trivial ($L_e = (\mathbf{I} - \mathbf{T})L$)

Conclusion / Take-away messages

Some properties looked obvious (to me!), but proved wrong:

- there is no uniform finite rank approximation of $\mathbf{T}$, and $\mathbf{T}_b$ (in scenes with edges)
- $\mathbf{T}_b$ is not invertible
- the cause of problems is not visibility ;-)
  - for $\mathbf{T}$, the cause is partial integration (see paper)
  - for $\mathbf{T}_b$, the cause is frequency preservation next to abutting edges

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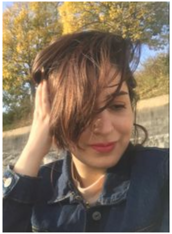
Cyril Soler



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